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SPLINE APPROXIMATIONS FOR

LINEAR NONAUTONOMOUS DELAY SYSTEMS

H. Thomas Banks

and

I. Gary Rosen

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INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING

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A semi-discrete approximation framework for linear nonautonomous nonhomogeneous functional differential equations of retarded type was developed. The approximation schemes were constructed and convergence results were obtained through the approximation of an associated abstract evolution operator. Computer implementation of the schemes is outlined and a spline based method included in the framework is constructed. Extensions of the semi-discrete methods to schemes incorporating full discretization and difference (CONTINUED)

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ITEM #20 CONTINUED: equation approximation are also discussed. Numerical results for several examples demonstrating the feasibility of the schemes are presented.

SPLINE APPROXIMATIONS FOR LINEAR

NONAUTONOMOUS DELAY SYSTEMS*

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ABSTRACT

We develop a semi-discrete approximation framework for linear non-autonomous nonhomogeneous functional differential equations of retarded type. The approximation schemes are constructed and convergence results are obtained through the approximation of an associated abstract evolution operator. Computer implementation of the schemes is outlined and a spline based method included in the framework is constructed. Extensions of the semi-discrete methods to schemes incorporating full discretization and difference equation approximation are also discussed. Numerical results for several examples demonstrating the feasibility of the schemes are presented.

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1. Introduction

Our presentation deals with spline based approximation theory for linear nonautonomous nonhomogeneous functional differential equations (FDE) via approximation of the associated evolution operator for the homogeneous equation. Our theoretical framework and the resulting convergence results are analogous to the Trotter-Kato type approach for convergence of solution semigroups for autonomous delay systems as developed in [8]. We restrict our discussions to piecewise linear spline approximations, but have attempted to state the theoretical results in a form so that one might, if one so desired, easily extend the ideas in a rather straight forward manner to treat higher order spline approximation schemes.

One might also view the discussions here as one concrete realization of the evolution operator theory presented for abstract nonautonomous equations in [14]. Indeed, as we shall see below our approach is very closely related to the efforts of Reber in [17, 18] who used the Krein approach to develop for nonautonomous FDE control problems an approximation theory based on the so-called "averaging" approximations of [4,5]. However, since the spline based methods of [8] have been shown to offer considerable improvement over the averaging schemes, and since the needs for schemes to treat nonautonomous problems are rather obvious in a number of areas of applications (for one example, see the discussion of the tracking models in [24]), we feel that our modest contribution below towards the development of spline schemes is warranted. We demonstrate the computational efficacy of our ideas by presenting a sample of our numerical findings in section 6.

Other authors have considered approximation schemes for nonautonomous delay systems. Both Delfour [11] and Reber [18] consider optimal control problems for linear nonautonomous retarded FDE and develop "full discretization" techniques (state and time discretization) that employ the averaging ideas of [4,5]. Kappel and Schappacher in [13] consider nonlinear nonautonomous delay systems which they approximate by linear interpolating spline schemes in the "state" space C. This results in essentially an averaging type scheme as opposed to the spline schemes discussed below. In a similar spirit Kunisch [15] discusses the averaging and essentially equivalent linear interpolating spline schemes for optimal control of neutral FDE with nonautonomous retarded terms.

It is possible to take other approaches to a spline approximation theory for delay systems. In [2] we combine dissipativeness of nonlinear operators with Gronwall estimates to develop one approach to approximation of nonlinear FDE. These ideas are pursued in [7, 10] to yield spline approximation methods (in the context of parameter estimation schemes) for quite general nonlinear nonautonomous systems of FDE. In addtition to differing in spirit from the approach of [7, 10], our results below are applicable to linear systems with nonsmooth coefficients while the theory of [7, 10] requires some smoothness on the coefficient matrices if it is applied to linear systems.

For our detailed development below we consider ordinary differential equation (ODE) or semi-discrete approximations to linear FDE. These ODE for the "Fourier" coefficients of the approximate solution relative to a fixed spline basis must then be solved by a high order ODE solver (a fourth order Runge-Kutta in the case of the examples presented in section 6). An alternative approach involving immediate full discretization (in both time and state) and resulting in a difference equation approximation of the FDE could be taken in the spirit of the efforts of Rosen in [19, 20]. We give a brief indication of some of these ideas in section 5 but will not pursue a full detailed development along these lines.

While we shall not discuss such applications directly, the reader should be aware that the approximation ideas developed here are readily (and profitably) used in optimal control and parameter estimation problems (e.g. see [1], [3], [5], [6], [9], [12], [16], and [21]).

The presentation below is organized in the following manner. In section 2 we summarize equivalence results between FDE and abstract evolution equations. We also establish a dissipative condition for the operators involved that is crucial in our development. Basic approximation results are given in section 3, first for systems with continuously differentiable coefficients and then for systems with L_∞ coefficient matrices. This is followed by section 4 in which we develop a particular piecewise linear spline scheme in detail and explain how it is to be implemented. A brief discussion of full discretization ideas is given in section 5. We conclude the paper with some representative numerical findings in section 6.

Notation throughout is completely standard with respect to symbols for C, L_p, etc. We denote the usual Sobolev spaces $W_2^{(k)}$ of functions f with $f^{(k-1)}$ absolutely continuous and $f^{(k)}$ in L_2 by H^k. For Lebesgue spaces of R^n-valued functions on (a,b) we adopt the notation $L_p^n(a,b)$ while $\mathcal{L}_{n\times n}$ denotes the space of n square matrices. Finally, we shall sometimes use $D\phi$ to represent the derivative of a function ϕ.

2. The Linear Non-autonomous Functional Differential Equation and its Equivalent Formulation as an Abstract Evolution Equation

In this section we describe the functional differential equation (FDE) initial value problem for which we seek approximation schemes and give an equivalent formulation of it as an abstract evolution equation in an infinite dimensional Hilbert space. Many of the results and ideas which are outlined and summarized below are discussed in detail in [18, sections 2 and 3].

We consider n-vector systems of the form

$$\dot{x}(t) = L(t)x_t + f(t) \qquad t \geq 0 , \tag{2.1}$$

$$x(s) = \eta \qquad x_s = \phi, \tag{2.2}$$

where $f \in L^n_{2,loc}(s,+\infty)$, $\eta \in R^n$, $\phi \in L^n_2(-r,0)$ and x_t denotes the function $\theta \to x(t+\theta)$, $-r \leq \theta \leq 0$. We assume that for each $t \geq 0$, the linear operator $L(t):L^n_2(-r,0) \to R^n$ has the form

$$L(t)\phi = \sum_{i=0}^{\nu} A_i(t)\phi(-\tau_i) + \int_{-r}^{0} A(t,\theta)\phi(\theta)d\theta, \tag{2.3}$$

with $0 = \tau_0 < \tau_1 < \tau_2 \cdots < \tau_\nu = r$, $A_i \in L_\infty([s,+\infty),\mathcal{L}_{n\times n})$, $i = 0,1,2 \cdots \nu$ and the function $t \to A(t,\cdot)$ an element of $L_\infty([s,+\infty), L_2([-r,0],\mathcal{L}_{n\times n}))$. The point evaluations of $\phi \in L^n_2(-r,0)$ required in the evaluation of $L(t)\phi$ pose no essential conceptual difficulties since, roughly speaking, we shall interpret solutions of (2.1) as functions satisfying that equation in integrated form (i.e. the differentiated form in the almost everywhere sense) and thus any occurance of $L(t)\phi$ for ϕ only an L_2 "function" can be considered as appearing under an integral with the $L(t)\phi$ then denoting an equivalence class of functions. For a further discussion of this point, we refer the reader to [8].

A solution to (2.1), (2.2) is a function $x:[s-r,T] \to R^n$, $T > 0$, such that $x \in H^1(s,T)$, x satisfies equation (2.1) almost everywhere in $[s,T]$, $x(s) = \eta$ and $x_s = \phi$. Standard arguments [17] can be used to show that the FDE initial value problem (2.1), (2.2) has a unique solution which depends continuously upon the initial conditions and the nonhomogeneous term f. We shall on occasion employ the notation $x(t;\eta,\phi,f)$ (and $x_t(\eta,\phi,f)$) in order to denote this unique solution (and its past history on $[t-r,t]$) to (2.1), (2.2) corresponding to a particular choice of η,ϕ, and f.

We next define the Hilbert space Z by

$$Z \equiv R^n \times L_2^n(-r,0),$$

with inner product

$$\langle\cdot,\cdot\rangle_Z = \langle\cdot,\cdot\rangle_{R^n} + \langle\cdot,\cdot\rangle_{L_2},$$

and reformulate the FDE initial value problem (2.1), (2.2) as an abstract evolution equation in Z. Corresponding to $f \equiv 0$, it is possible to define a solution operator for (2.1), (2.2) on Z by

$$U(t,s)(\eta,\phi) = (x(t;\eta,\phi,0),x_t(\eta,\phi,0)).$$

It is easily verified that for $T > s$ the operators $\{U(t,s):t \in [s,T]\}$ are continuous in t and uniformly bounded. In addition $U(t,s)$ is an evolution operator on Z in that the uniqueness of solutions to (2.1), (2.2) guarantees that it satisfies $U(s,s) = I$ and the transition property $U(t,s) = U(t,\tau)U(\tau,s)$ for all $s \leq \tau \leq t \leq T$.

Returning to the nonhomogeneous problem, we define for arbitrary $f \in L^n_{2,\mathrm{loc}}(s,+\infty)$

$$z(t,s;\eta,\phi,f) = U(t,s)(\eta,\phi) + \int_s^t U(t,s)(f(s),0)ds. \tag{2.4}$$

For each $(\eta,\phi) \in Z$, $f \in L^n_{2,\mathrm{loc}}(s,+\infty)$ and $t \geq s$ the expression given in (2.4) exists and is continuous in t. Furthermore, it can be shown that

$$z(t,s;\eta,\phi,f) = (x(t;\eta,\phi,f),x_t(\eta,\phi,f)). \tag{2.5}$$

Equation (2.5) states that (2.4) and (2.1), (2.2) are equivalent, and in fact, forms the basis for the approximation schemes developed in the next section. Indeed, we construct convergent approximations to the solution of (2.1), (2.2) via the construction of convergent approximations to z of (2.4).

If one is willing to impose additional restrictions on (2.1), (2.2) a stronger result can be established. Consider the initial value problem (2.1), (2.2) with the additional assumption that the coefficient matrices, the kernel in the distributed term and the nonhomogeneous term be continuously differentiable in t and that $(\eta,\phi) \in W = \{(\eta,\phi) \in Z : \phi \in H^1(-r,0), \eta = \phi(0)\}$. It then can be shown that z given in (2.4) is the unique solution to the abstract evolution equation in Z given by

$$\dot{z}(t) = A(t)z(t) + (f(t),0), \tag{2.6}$$

$$z(s) = (\eta,\phi), \tag{2.7}$$

where for each $t > s$ the operators $A(t): W \subset Z \to Z$ are defined by

$$A(t)(\phi(0),\phi) = (L(t)\phi,\dot{\phi}). \tag{2.8}$$

In addition it can be verified that

$$w(t) = (x(t;\eta,\phi,f),\ x_t(\eta,\phi,f)),$$

is also a solution to (2.6), (2.7) and hence must coincide with z. Thus under these stronger hypotheses (2.1) - (2.2), (2.4), and (2.6) - (2.7) are all equivalent.

The existence of an inner product on Z, equivalent to the standard inner product on Z defined above, and an ω for which the operator $A(t) - \omega I$ is dissipative plays an essential role in many of the arguments which follow. Toward this end, we define the same inner product on Z as that one employed in [8] and [10] for similar purposes. Let ρ be the step function on $[-r,0]$ defined by

$$\rho(\theta) = j \qquad -\tau_{\nu-j+1} \leq \theta < -\tau_{\nu-j}, \qquad j = 1,2,\cdots \nu,$$

and Z_ρ the space Z with inner product $<\cdot,\cdot>_\rho$ are given by

$$<(\xi,\psi),(\eta,\phi)>_\rho \;=\; \xi^T\eta + \int_{-r}^{0} \psi(\theta)\phi(\theta)\rho(\theta)d\theta \;.$$

It is easily verified that the $<\cdot,\cdot>_\rho$ inner product is equivalent to the standard inner product on Z and moreover, the following lemma can be established.

LEMMA 2.1. For each $t > s$, $A(t) - \omega I$ is dissipative in Z_ρ. That is

$$<A(t)z,z>_\rho \leq \omega<z,z>_\rho,$$

for each $z \in W$ with

$$\omega = \frac{3}{2}\, m_1 + \frac{\nu}{2}\, m_1^2 + \frac{\nu}{2} + 1$$

and

$$m_1 = \sum_{i=0}^{\nu} |A_i|_\infty + |A|_\infty \;.$$

It is in fact the case that $A(t) - \omega I$ is maximal dissipative. That is to say, $A(t) - \omega I$ is onto. The verification of this latter claim can be found in [18, section 3]. The reader should note that while the hypothesis of Lemma 3.3 in [18] include the assumption of smooth coefficients on the right hand side of the FDE, it is easily seen that this assumption does not play a role in the arguments used to show that $A(t) - \omega I$ is onto.

Remark 2.1. By the Lumer-Phillips theorem (cf. [25, section IX.8]) the fact that for $t \in (s,+\infty)$ fixed, $A(t) - \omega I$ is a maximal dissipative operator on the Hilbert space Z is sufficient to conclude that $\mathcal{A}(t)$ is the infinitesimal generator of a C_0 semigroup of bounded linear operators on Z.

3. Approximation Results

Following the ideas discussed in [8] we base our approximation schemes on the following construction. For each $N = 1,2,\cdots,\{Z_N, P_N, \mathcal{A}_N(t)\}$ will be called an approximation scheme for the initial value problem (2.1), (2.2) if $\{Z_N\}$ is a sequence of finite dimensional subspaces of Z_ρ, $\{P_N\}$ is a sequence of operators, where for each N, $P_N: Z_\rho \to Z_N$ is the orthogonal projection from Z_ρ onto Z_N and $\{\mathcal{A}_N(t)\}$ is a sequence of t-dependent operators on Z_N.

THEOREM 3.1. Consider the FDE initial value problem (2.1), (2.2) under the additional assumption that $A_i \in C^1([s,+\infty),\mathcal{L}_{n\times n})$, $t \to A(t,\cdot) \in C^1([s,+\infty), L_2((-r,0),\mathcal{L}_{n\times n}))$ and suppose that $\{Z_N, P_N, \mathcal{A}_N(t)\}$ is an approximation scheme for (2.1), (2.2) satisfying the following conditions

(1) $Z_N \subset W = \mathrm{dom}(\mathcal{A}(t))$ $N = 1,2,\cdots$.

(2) For each $t \geq s$, $\mathcal{A}_N(t): Z_N \to Z_N$ is defined by $\mathcal{A}_N(t) = P_N \mathcal{A}(t)$, $N = 1,2\cdots$.

(3a) $\lim_{N\to\infty} P_N z = z$ in Z for all $z \in Z$.

(3b) For $\hat{\psi} \in W$ with $P_N\hat{\psi} = P_N(\psi(0),\psi) = (\psi_N(0),\psi_N)$ we have $\lim_{N\to\infty} L(t)\psi_N = L(t)\psi$ in R^n for each $t \in [s,T]$ and $\lim_{N\to\infty} D\psi_N = D\psi$ in $L_2^n(-r,0)$ with $|D(\psi_N - \psi)| \leq K|D^2\psi|$, K independent of N and ψ for all $\hat{\psi} \in W$ with $\psi \in H^2(-r,0)$.

Then if $U_N(t,s)$ denotes the evolution operator (fundamental matrix solution) corresponding to the finite dimensional ordinary differential equation in Z_N given by

$$\dot{z}_N(t) = A_N(t)z_N(t),$$

we have that

$$\lim_{N\to\infty} |[P_N U(t,s) - U_N(t,s)P_N]z_0| = 0,$$

for each $z_0 \in Z$, uniformly in t for $t \in [s,T]$.

Proof. An application of Lemma 2.1 and the fact that the P_N are orthogonal projections yields the following: For $z_N \in Z_N$

$$\begin{aligned}\langle A_N(t)z_N, z_N\rangle_\rho = \langle P_N A(t)z_N, z_N\rangle_\rho &= \langle A(t)z_N, P_N z_N\rangle_\rho \\ &= \langle A(t)z_N, z_N\rangle_\rho \le \omega \langle z_N, z_N\rangle_\rho,\end{aligned}$$

where ω is defined in the statement of Lemma 2.1, and is independent of N. The calculation above, and the fact that $A_N(t)$ is a bounded linear operator defined on the finite dimensional space Z_N are sufficient to guarantee that $A_N(t) - \omega I$, $N = 1,2,\cdots$, are maximal dissipative, and moreover

$$\sigma(A_N(t)) \subset \{\lambda \in ¢ : \mathrm{Re}\lambda < \omega\}, \qquad N = 1,2,\cdots,\ t \ge s.$$

Thus for all $\lambda \in ¢$ with $\mathrm{Re}\,\lambda \ge \omega$ the resolvent operators $R(\lambda, A_N(t))$ exist and by standard arguments (c.f. [14, p. 85]) we have

$$(3.1)\quad |R(\lambda, A_N(t))|_\rho = |(A_N(t) - \lambda I)^{-1}|_\rho \le \frac{1}{\mathrm{Re}\lambda - \omega} \qquad N = 1,2,\cdots, \quad t \in [s,+\infty).$$

Inequality (3.1) and the same arguments used to establish the validity of Theorem 3.5 in [18] allows us to conclude

$$|U_N(t,s)|_Z \le Me^{\omega(t-s)},$$

where M is a constant independent of N and $L(\cdot)$, the homogeneous part of the right hand side of the FDE (2.1). The reader is instructed to note that while the constant M derived in the proof of Theorem 3.5 in [18] does depend on $L(\cdot)$, our M does not. This is a consequence of the fact that the weighting function ρ in the inner product $\langle\cdot,\cdot\rangle_\rho$ is independent of the A_i, whereas that in the inner product $\langle\cdot,\cdot\rangle_\lambda$ chosen by Reber is not. The significance of our choice for the inner product will be apparent in the discussions below pertaining to extensions of our convergence results to equations with the A_i, A only in L_∞.

Let D be the subset of Z defined by

$$D = \{(\eta,\phi) \in Z : \phi \in C^2(-r,0), \eta = \phi(0), L(s)\phi = \dot{\phi}(0)\}.$$

Using the fact that $\mathcal{A}(s)$ is the infinitesimal generator of a C_0 semigroup of operators on Z (cf. Remark 2.1) arguments similar to those used to verify Lemma 2.2 of [8] can be used to establish the fact that D is dense in Z. Consider next the initial value problem in Z given by

$$\dot{z}(t) = \mathcal{A}(t)z(t), \tag{3.2}$$

$$z(s) = z_0, \qquad z_0 \in D, \tag{3.3}$$

and the following identity derived from it:

$$P_N\dot{z}(t) = \mathcal{A}_N(t)P_Nz(t) + [P_N\mathcal{A}(t)z(t) - \mathcal{A}_N(t)P_Nz(t)].$$

Recalling that the P_N are orthogonal projections, and therefore uniformly bounded, we have

$$\frac{d}{dt}[P_Nz(t)] = \mathcal{A}_N(t)[P_Nz(t)] + [P_N\mathcal{A}(t)z(t) - \mathcal{A}_N(t)P_Nz(t)],$$

and thus, by the variation of constants formula

$$(3.4)\qquad P_N z(t) = U_N(t,s)P_N z(s) + \int_s^t U_N(t,\tau)[P_N A(\tau) - A_N(\tau)P_N]z(\tau)d\tau .$$

Since $z_0 \in D \subset W$, the unique solution z to (3.2), (3.3) is given by

$$z(t) = U(t,s)z_0,$$

and hence (3.4) can be rewritten as

$$P_N U(t,s)z_0 - U_N(t,s)P_N z_0 = \int_s^t U_N(t,\tau)[P_N A(\tau) - A_N(\tau)P_N]z(\tau)d\tau .$$

Therefore

$$\begin{aligned}(3.5)\qquad |P_N U(t,s)z_0 - U_N(t,s)P_N z_0| &= \left| \int_s^t U_N(t,\tau)[P_N A(\tau) - A_N(\tau)P_N]z(\tau)d\tau \right| \\ &\le \int_s^t |U_N(t,\tau)|\,|[P_N A(\tau) - A_N(\tau)P_N]z(\tau)|d\tau \\ &\le Me^{\omega(T-s)} \int_s^T |[P_N A(\tau) - A_N(\tau)P_N]z(\tau)|d\tau .\end{aligned}$$

Let $z = (\phi(0),\phi) \in Z$ be such that $\phi \in H^2(-r,0)$.

Then

$$\begin{aligned}|[P_N A(t) - A_N(t)P_N]z|^2 &= |[P_N A(t) - P_N A(t)P_N](\phi(0),\phi)|^2 \\ &\le |[A(t) - A(t)P_N](\phi(0),\phi)|^2 \\ &= |(L(t)\phi, D\phi) - (L(t)\phi_N, D\phi_N)|^2 \\ &= |(L(t)\phi - L(t)\phi_N)|^2_{R^n} + |D\phi - D\phi_N|^2_{L_2} \to 0 ,\end{aligned}$$

as $N \to \infty$ for each $t \in [s,T]$ by condition (3b). Thus $P_N A(t)z - A_N(t)P_N z \to 0$ for each $t \in [s,T]$, and each $z = (\phi(0),\phi) \in Z$ with $\phi \in H^2(-r,0)$.

Now under the smoothness assumption on $L(\cdot)$ for $z_0 = (\phi(0),\phi) \in D$ we have that $x(\cdot;\phi(0),\phi,0) \in H^2(s,+\infty)$. Consequently $x_t(\phi(0),\phi,0) \in H^2(-r,0)$ for $t \geq s$. Therefore in the light of the equivalence established in section 2, and the arguments above, we have

$$(3.6) \quad |[P_N A(t) - A_N(t)P_N]z(t)| = |[P_N A(t) - A_N(t)P_N]U(t,s)z_0|$$

$$= |[P_N A(t) - A_N(t)P_N](x(t;\phi(0),\phi,0),x_t(\phi(0),\phi,0))|$$

$$\to 0 \text{ for each } t \in [s,T] \text{ with the convergence}$$

being dominated. Indeed, $x \in H^2(s,+\infty)$ and an application of condition (3b) yields

$$|[P_N A(t) - A_N(t)P_N]z(t)|^2 \leq |[A(t) - A(t)P_N](x(t;\phi(0),\phi,0),x_t(\phi(0),\phi,0))|^2$$

$$= |(L(t)x_t(\phi(0),\phi,0),Dx_t(\phi(0),\phi,0))$$

$$- (L(t)x_t^N(\phi(0),\phi,0),Dx_t^N(\phi(0),\phi,0))|^2$$

$$= |L(t)(x_t(\phi(0),\phi,0) - x_t^N(\phi(0),\phi,0))|^2_{R^n}$$

$$+ |D(x_t(\phi(0),\phi,0) - x_t^N(\phi(0),\phi,0))|^2_2$$

$$\leq |L(t)(x_t(\phi(0),\phi,0) - x_t^N(\phi(0),\phi,0))|^2_{R^n} + K^2|D^2 x_t(\phi(0),\phi,$$

$$\leq |L(t)(x_t(\phi(0),\phi,0) - x_t^N(\phi(0),\phi,0))|^2_{R^n} + K^2|D^2 x|^2_{L_2^n(s-r,}$$

where the bound on the second term in the above estimates is a consequence of condition (3b) and $(x^N(t),x_t^N) \equiv P_N(x(t),x_t)$. We note that it is not true in general that $x_t^N = (x^N)_t$. To see that the convergence in the first term is dominated as well, we first observe that $x^N(t) \to x(t)$ uniformly in t for $t \in [s-r,T]$. This follows from condition (3a) and the fact that the subset of Z given by $\{z(t) : t \in [s,T]\}$ is compact (being the continuous image of a compact set in R). Then, since for $-r \leq \theta \leq 0$

$$x_t^N(\theta) = x_t^N(0) + \int_0^\theta (Dx_t^N)(\sigma)d\sigma,$$

we have

$$\begin{aligned}|x_t^N(\theta) - x_t(\theta)| &\leq |x_t^N(0) - x_t(0)| + \int_\theta^0 |D(x_t^N - x_t)(\sigma)|d\sigma \\ &\leq |x_t^N(0) - x_t(0)| + r^{\frac{1}{2}}|D(x_t^N - x_t)|_2 \\ &\leq |x^N(t) - x(t)| + r^{\frac{1}{2}}K|D^2x_t|_2 \\ &\leq |x^N(t) - x(t)| + r^{\frac{1}{2}}K|D^2x|_{L_2^n(s-r,T)} \\ &\leq |x^N - x|_\infty + r^{\frac{1}{2}}K|D^2x|_{L_2^n(s-r,T)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left|L(t)\left(x_t(\phi(0),\phi,0) - x_t^N(\phi(0),\phi,0)\right)\right|_{R^n} \\ &\leq \left(|x^N(\cdot;\phi(0),\phi,0) - x(\cdot;\phi(0),\phi,0)|_\infty + r^{\frac{1}{2}}K|D^2x|_{L_2^n(s-r,T)}\right) \\ &\cdot \left(\sum_{j=0}^{\nu}|A_j(\cdot)|_\infty + r^{\frac{1}{2}}\left|\left(\int_{-r}^0 |A(\cdot,\theta)|^2 d\theta\right)^{\frac{1}{2}}\right|_\infty\right).\end{aligned}$$

Returning to (3.5), the convergence stated in (3.6) together with the dominated convergence theorem allow us to conclude that for each $z_0 \in D$ $|P_N U(t,s)z_0 - U_N(t,s)P_N z_0| \to 0$ as $N \to \infty$, uniformly in t for $t \in [s,T]$. But D is a dense subset of Z and the operators

$$[P_N U(t,s) - U_N(t,s)P_N] ,$$

are uniformly bounded in N. Therefore

$$|P_N U(t,s)z - U_N(t,s)P_N z| \to 0, \quad N \to \infty,$$

uniformly in t for $t \in [s,T]$ for each $z \in Z$.

Remark. It is possible to obtain estimates for the rate of convergence of the evolution operators on restricted classes of initial data. The relevant arguments are in the same vein as those used to derive the estimates for the rate of convergence in the autonomous case. The details of these arguments can be found in [8].

Turning our attention to the nonhomogeneous problem, for $(\eta,\phi) \in Z$ and $f \in L^n_{2,loc}(s,+\infty)$ we consider the approximating finite dimensional nonhomogeneous ordinary differential equation in Z_N given by

$$\dot{z}_N(t) = A_N(t)z_N(t) + P_N(f(t),0),$$
$$z_N(s) = P_N(\eta,\phi).$$

Using the variation of constants formula, we can write down the solution to this initial value problem. It is given by $z_N(t,s;\eta,\phi,f) = U_N(t,s)P_N(\eta,\phi) + \int_s^t U_N(t,\sigma)P_N(f(\sigma),0)d\sigma$ for $t \geq s$. An application of arguments analogous to those used to verify Theorem 3.2 of [8] will establish the validity of the following theorem.

THEOREM 3.2. Under the hypotheses and conditions of Theorem 3.1 we have

(a) For $(\eta,\phi) \in Z$ and $f \in L^n_2(s,T)$

$$\lim_{N\to\infty} z_N(t,s;\eta,\phi,f) = z(t,s;\eta,\phi,f),$$

uniformly in t for $t \in [s,T]$ and uniformly in f for f in bounded subsets of $L_2^n(s,T)$.

(b) For $(x_N(t),y_N(t)) \in Z_N$ defined by $(x_N(t),y_N(t)) = z_N(t,s;\eta,\phi,f)$, we have

$$\lim_{N\to\infty} x_N(t) = x(t;\eta,\phi,f),$$

uniformly in t for $t \in [s,T]$.

(c) If $\{f_k\}$ is a sequence in $L_2^n(s,T)$ converging weakly to f then

$$\lim_{N,k\to\infty} z_N(t,s;\eta,\phi,f_k) = z(t,s;\eta,\phi,f),$$

uniformly in t for $t \in [s,T]$.

Remark. Although it will not be discussed in this paper, part (c) of Theorem 3.2 above plays an essential role in the application of our approximation results to the solution of optimal control problems governed by FDE of the form (2.1). These ideas are discussed in detail for the case of an autonomous equation in [5].

We conclude this section with a discussion of the details involved in extending the approximation results above to FDE initial value problems of the form (2.1), (2.2) with non-smooth right hand sides. Define

$$\Lambda = L_\infty\Big((s,+\infty),\Big(\mathop{\times}_{0}^{\nu} \mathcal{L}_{n\times n}\Big) \times L_2\big((-r,0),\mathcal{L}_{n\times n}\big)\Big),$$

$$\Lambda_c = C^1\Big((s,+\infty),\Big(\mathop{\times}_{0}^{\nu} \mathcal{L}_{n\times n}\Big) \times L_2\big((-r,0),\mathcal{L}_{n\times n}\big)\Big).$$

Then $\Lambda_c \subset \Lambda$ and for $\lambda = (A_0,A_1,\cdots,A_\nu,A) = (A_0(\cdot,\lambda),A_1(\cdot,\lambda),\cdots,$ $A_\nu(\cdot,\lambda),A(\cdot,\cdot,\lambda))$, an element of Λ or Λ_c define

$$|\lambda|_\infty = \sum_{j=0}^{\nu} |A_j|_\infty + |(\int_{-r}^{0} |A(\cdot,\theta)|^2 d\theta)^{\frac{1}{2}}|_\infty .$$

Let $L(t;\lambda)$ be the operator defined in (2.3) where the coefficient matrices are the components of λ. Let $U_\lambda(t,s)$ denote the solution operator and $x(\cdot,\eta,\phi,0,\lambda)$ the solution of the FDE initial value problem (2.1), (2.2) with $f \equiv 0$ and $L(t) = L(t;\lambda)$. Let $A(t;\lambda):W \subset Z \to Z$ denote the operator defined in (2.8) with $L(t) = L(t;\lambda)$ and $U_{\lambda,N}(t,s)$ denote the solution operator for the approximating initial value problem in Z_N:

$$\dot{z}_N(t) = A_N(t;\lambda)z_N(t),$$

$$z_N(s) = z_N^0$$

with

$$A_N(t;\lambda) = P_N A(t;\lambda).$$

<u>LEMMA 3.1</u>. Let $\lambda \in \Lambda$ be fixed. Then there exists a sequence $\{\lambda_k\} \subset \Lambda_c$ such that $|\lambda_k - \lambda|_\infty \to 0$, $k \to \infty$. Moreover, for $z_0 = (\phi(0),\phi) \in W$ we have

$$|U_{\lambda_k}(t,s)z_0 - U_\lambda(t,s)z_0|_Z^2 \le K|\lambda_k - \lambda|_\infty\ |z_0|$$

where $K = K(\lambda)$ is a constant independent of k and z_0.

Proof. The existence of the sequence $\{\lambda_k\} \subset \Lambda_c$ with $\lambda_k \to \lambda$ is a consequence of the fact that Λ_c is a dense subset of Λ. Next, we let

(3.7) $\quad z(t) = U_\lambda(t,s)z_0 = (x(t;\phi(0),\phi,0,\lambda),x_t(\phi(0),\phi,0,\lambda)),$

(3.8) $\quad z_k(t) = U_{\lambda_k}(t,s)z_0 = (x(t;\phi(0),\phi,0,\lambda_k),x_t(\phi(0),\phi,0,\lambda_k)),$

where the extreme right hand equalities in (3.7) and (3.8) follow from the equivalence established in section 2. We note that for each $t \ge s$, $z(t),z_k(t) \in W$.

By [18, Theorem 3.15], for $\lambda_k \in \Lambda_c$ and $z_0 \in W$ we can write

$$z_k(t) = z_0 + \int_s^t A(\sigma;\lambda_k) z_k(\sigma) d\sigma ,$$

and by [10, p. 23] for $\lambda \in \Lambda$ and $z_0 \in W$

$$z(t) = z_0 + \int_s^t A(\sigma;\lambda) z(\sigma) d\sigma.$$

Let $\Delta(t) = z(t) - z_k(t)$. Then $\Delta(t) \in W$ and

$$\begin{aligned}
(3.9) \qquad \Delta(t) &= \int_s^t [A(\sigma;\lambda) z(\sigma) - A(\sigma;\lambda_k) z_k(\sigma)] d\sigma \\
&= \int_s^t \{A(\sigma;\lambda)[z(\sigma) - z_k(\sigma)] + [A(\sigma;\lambda) - A(\sigma;\lambda_k)] z_k(\sigma)\} d\sigma \\
&= \int_s^t [A(\sigma;\lambda)\Delta(\sigma) + \delta A_k(\sigma) z_k(\sigma)] d\sigma,
\end{aligned}$$

where

$$\delta A_k(\sigma) \equiv A(\sigma;\lambda) - A(\sigma;\lambda_k).$$

If we then apply [2, Lemma 2.1] (3.9) implies

$$\begin{aligned}
|\Delta(t)|_\rho^2 = |\Delta(s)|_\rho^2 + 2 \int_s^t \{&\langle A(\sigma;\lambda)\Delta(\sigma), \Delta(\sigma)\rangle_\rho \\
&+ \langle \delta A_k(\sigma) z_k(\sigma), \Delta(\sigma)\rangle_\rho\} d\sigma.
\end{aligned}$$

But $\Delta(s) \equiv 0$ and (c.f. Lemma 2.1)

$$\langle A(\sigma;\lambda)\Delta(\sigma), \Delta(\sigma)\rangle_\rho \le \omega(\lambda) |\Delta(\sigma)|_\rho^2.$$

Therefore, an application of the Gronwall inequality yields

$$\begin{aligned}
|\Delta(t)|^2 &\le 2 \int_s^t \{\omega(\lambda)|\Delta(\sigma)|^2 + \tfrac{1}{2}|\delta A_k(\sigma) z_k(\sigma)|^2 + \tfrac{1}{2}|\Delta(\sigma)|^2\} d\sigma \\
&= \int_s^t (2\omega(\lambda) + 1)|\Delta(\sigma)|^2 d\sigma + \int_s^t |\delta A_k(\sigma) z_k(\sigma)|^2 d\sigma, \\
&\le \int_s^t |\delta A_k(\sigma) z_k(\sigma)|^2 d\sigma \;\; e^{(2\omega(\lambda)+1)(t-s)} .
\end{aligned}$$

Consider the integrand in the last expression above:

$$
\begin{aligned}
|\delta A_k(\sigma)z_k(\sigma)|^2 &= |[A(\sigma;\lambda) - A(\sigma;\lambda_k)]\big(x(\sigma;\phi(0),\phi,0,\lambda_k),x_\sigma(\phi(0),\phi,0,\lambda_k)\big)|^2 \\
&= |\big([L(\sigma;\lambda) - L(\sigma;\lambda_k)]x_\sigma(\phi(0),\phi,0,\lambda_k),0\big)|^2 \\
&= |\sum_{i=0}^{\nu}[A_i(\sigma;\lambda) - A_i(\sigma;\lambda_k)]x(\sigma-\tau_i;\phi(0),\phi,0,\lambda_k) \\
&\quad + \int_{-r}^{0}[A(\sigma,\theta;\lambda) - A(\sigma,\theta;\lambda_k)]x(\sigma+\theta;\phi(0),\phi,0,\lambda_k)\,d\theta|^2 .
\end{aligned}
$$

Recalling that for $\lambda_k \in \Lambda_c$ and $z_0 \in W$

$$|U_{\lambda_k}(t,s)z_0| \leq M|z_0|e^{\omega(\lambda_k)(t-s)},$$

by (3.8) we have

$$|x(\tau;\phi(0),\phi,0,\lambda_k)| \leq M|z_0|e^{\omega(\lambda_k)\tau}, \quad s-r \leq \tau \leq T,$$

and hence

$$|x(\cdot;\phi(0),\phi,0,\lambda_k)| \leq M|z_0|e^{\omega(\lambda_k)T} \leq \hat{K}|z_0| ,$$

where $\hat{K}$ is independent of k for all k sufficiently large. Therefore for $\sigma \in [s,T]$

$$
\begin{aligned}
|\delta A_k(\sigma)z_k(\sigma)|^2 &\leq \Big(\hat{K}|z_0|\Big[\sum_{i=0}^{\nu}|A_i(\cdot;\lambda) - A_i(\cdot;\lambda_k)|_\infty \\
&\quad + |\int_{-r}^{0}(A(\cdot,\theta;\lambda) - A(\cdot,\theta;\lambda_k))d\theta|_\infty\Big]\Big)^2 \\
&\leq \Big(\hat{K}|z_0|\Big[\sum_{i=0}^{\nu}|A_i(\cdot;\lambda) - A_i(\cdot;\lambda_k)|_\infty \\
&\quad + r^{\frac{1}{2}}|\Big(\int_{-r}^{0}|A(\cdot,\theta;\lambda) - A(\cdot,\theta;\lambda_k)|^2 d\theta\Big)^{\frac{1}{2}}|_\infty\Big]\Big)^2
\end{aligned}
$$

$$\leq \tilde{K}^2 |z_0|^2 \left(\sum_{i=0}^{\nu} |A_i(\cdot;\lambda) - A_i(\cdot;\lambda_k)|_\infty \right.$$

$$\left. + \left| \left(\int_{-r}^{0} |A(\cdot,\theta;\lambda) - A(\cdot,\theta;\lambda_k)|^2 d\theta \right)^{\frac{1}{2}} \right|_\infty \right)^2$$

$$= \tilde{K}^2 |z_0|^2 \; |\lambda - \lambda_k|_\infty^2 .$$

Thus, we have

$$|U_\lambda(t,s)z_0 - U_{\lambda_k}(t,s)z_0|^2 = |z(t) - z_k(t)|^2 = |\Delta(t)|^2$$

$$\leq \left\{ \int_s^t |\delta A_k(\sigma) z_k(\sigma)|^2 d\sigma \right\} e^{(2\omega(\lambda)+1)(t-s)}$$

$$\leq \left\{ \int_s^t \tilde{K}^2 |z_0|^2 |\lambda - \lambda_k|_\infty^2 d\sigma \right\} e^{2(\omega(\lambda)+1)(t-s)}$$

$$\leq (T-s)\tilde{K}^2 |z_0|^2 |\lambda - \lambda_k|_\infty^2 \; e^{2(\omega(\lambda)+1)(T-s)},$$

which implies

$$|U_\lambda(t,s)z_0 - U_{\lambda_k}(t,s)z_0| \leq K|z_0| \; |\lambda - \lambda_k|_\infty ,$$

where

$$K = \tilde{K}(T-s)^{\frac{1}{2}} e^{(\omega(\lambda)+1)(T-s)} .$$

Let $\lambda = (A_0, A_1, \cdots, A_\nu, A) \in \Lambda$ be given and consider the FDE initial value problem

$$\dot{x}(t) = L(t;\lambda)x_t,$$

$$(x(s), x_s) = z_0 = (\eta, \phi).$$

Using the fact that W is dense in Z and Λ_c is dense in Λ, for $\varepsilon > 0$ given, we can choose $\tilde{z}_0 \in W$ such that

$$|z_0 - \tilde{z}_0| < \frac{\varepsilon}{M} e^{-\omega(\lambda)(T-s)} ,$$

and $\tilde{\lambda} \in \Lambda_c$ such that

$$|\lambda-\tilde{\lambda}|_\infty < \frac{\varepsilon}{K(\lambda)(\frac{\varepsilon}{M} e^{-\omega(\lambda)(T-s)} + |z_0|)} .$$

Since

$$|\tilde{z}_0| \le |\tilde{z}_0 - z_0| + |z_0| \le \frac{\varepsilon}{M} e^{-\omega(\lambda)(T-s)} + |z_0| ,$$

we have

$$|\lambda-\tilde{\lambda}|_\infty < \frac{\varepsilon}{K(\lambda)|\tilde{z}_0|} .$$

Now,

$$\begin{aligned}
(3.10)\quad |[P_N U_\lambda(t,s) - U_{\tilde{\lambda},N}(t,s)P_N]z_0| &\le |P_N U_\lambda(t,s)[z_0 - \tilde{z}_0]| \\
&+ |[P_N U_\lambda(t,s) - P_N U_{\tilde{\lambda}}(t,s)]\tilde{z}_0| + |[P_N U_{\tilde{\lambda}}(t,s) - U_{\tilde{\lambda},N}(t,s)P_N]\tilde{z}_0| \\
&+ |U_{\tilde{\lambda},N}(t,s)P_N[\tilde{z}_0 - z_0]| \le Me^{\omega(\lambda)(T-s)}|z - \tilde{z}_0| + K(\lambda)|\tilde{z}_0|\ |\lambda-\tilde{\lambda}|_\infty \\
&+ |[P_N U_{\tilde{\lambda}}(t,s) - U_{\tilde{\lambda},N}(t,s)P_N]\tilde{z}_0| + M\ |z-\tilde{z}_0|e^{\omega(\tilde{\lambda})(T-s)} \\
&\le Me^{\omega(\lambda)(T-s)} \frac{\varepsilon}{M} e^{-\omega(\lambda)(T-s)} + K(\lambda)|\tilde{z}_0| \frac{\varepsilon}{K(\lambda)|\tilde{z}_0|} \\
&+ |[P_N U_{\tilde{\lambda}}(t,s) - U_{\tilde{\lambda},N}(t,s)P_N]\tilde{z}_0| + Me^{\omega(\tilde{\lambda})(T-s)} \frac{\varepsilon}{M} e^{-\omega(\lambda)(T-s)} \\
&= 2\varepsilon + |[P_N U_{\tilde{\lambda}}(t,s) - U_{\tilde{\lambda},N}(t,s)P_N]\tilde{z}_0| + \varepsilon\, e^{(\omega(\lambda) - \omega(\tilde{\lambda}))(T-s)} \\
&\to 0 \text{ as } \varepsilon \to 0,
\end{aligned}$$

where the second term in the final expression above tends toward zero uniformly in t for $t \in [s,T]$ by Theorem 3.1 and the coefficient of the third term tends toward 1 as $\varepsilon \to 0$ as a consequence of the continuity of ω with respect to λ.

To summarize, (3.10) above reveals that in order to obtain an approximate solution to (2.1), (2.2) corresponding to $\lambda \in \Lambda$ it suffices to apply our approximation schemes to an approximating FDE initial value problem. That is, apply them to (2.1), (2.2) corresponding to $\tilde{\lambda} \in \Lambda_c$ a smooth approximation to λ. However, when actually implemented in practice, the approximation schemes which we have developed rely upon the application of standard discrete numerical methods for ordinary differential equations to the initial value problem

$$\dot{z}_N(t) = A_N(t;\lambda) z_N(t) \tag{3.11}$$

$$z_N(s) = P_N z_0 \ . \tag{3.12}$$

If we replace $A_N(t;\lambda)$ with $A_N(t;\tilde{\lambda})$ and if the time step in the ordinary differential equation integration is chosen sufficiently small, the resulting numerical solution would be indistinguishable from the one obtained by simply integrating (3.11), (3.12) as it stands. Thus, although the convergence result stated in Theorem 3.1 applies only to FDE initial value problems with smooth coefficients, in practice our approximation schemes are applicable to FDE with right hand sides in L_∞ as well.

4. Spline Approximations and Their Implementation

In this section we outline the ideas involved in realizing the approximation schemes discussed in section 3 in a manner appropriate for computer implementation. The formulation employed in [8] for schemes developed for autonomous equations can be modified so as to be applicable to schemes for nonautonomous equations as well. Therefore, with the exception of the modifications required by the time dependence of the operator $A(t)$, the results below are a summary of results found in [8]. We conclude the section with the construction of a particular realization using spline functions which satisfy the hypotheses and conditions of Theorem 3.1.

Let $\{Z_N, P_N, A_N(t)\}$ be an approximation scheme for the FDE initial value problem (2.1), (2.2) which satisfies the hypotheses and conditions of Theorem 3.1. Assume $\dim Z_N = k_N < \infty$ $N = 1,2,\cdots$. We recall condition (1) of Theorem 3.1, which states that $Z_N \subset W$, and fix a basis for Z_N, $\hat{\beta}_j^N = (\beta_j^N(0), \beta_j^N)$ $j = 1,2,\cdots, k_N$, with $\beta_j^N \in H^1(-r,0)$. Let β^N denote the $n \times k_N$ matrix function defined on $(-r,0)$ by

$$\beta^N = (\beta_1^N, \beta_2^N, \cdots, \beta_{k_N}^N),$$

and let

$$\hat{\beta}^N = (\beta^N(0), \beta^N).$$

For any $z_N \in Z_N$, we can write

$$z_N = \hat{\beta}^N \alpha_N = (\beta^N(0)\alpha_N, \beta^N \alpha_N),$$

where $\alpha_N \in R^{k_N}$ is the coordinate vector representation of z_N with respect to the basis $\{\hat{\beta}_j^N\}_{j=1}^{k_N}$.

Central to our approximation schemes is the finite dimensional approximating ordinary differential equation initial value problem in Z_N given by

$$\dot{z}_N(t) = A_N(t)z_N(t) + P_N(f(t),0), \tag{4.1}$$

$$z_N(s) = P_N(\eta,\phi). \tag{4.2}$$

If we let $w_N(t)$, $F_N(t)$, and w_N^s denote coordinate vector representation of $z_N(t)$, $P_N(f(t),0)$, and $P_N(\eta,\phi)$ respectively, and if we let $A_N(t)$ denote the matrix representation for the operator $A_N(t)$, all with respect to the basis $\{\hat{\beta}_j^N\}_{j=1}^{k_N}$, the initial value problem (4.1), (4.2) reduces to

$$\dot{w}_N(t) = A_N(t)w_N(t) + F_N(t)$$

$$w_N(s) = w_N^s,$$

which can then be solved via standard numerical methods for the integration of ordinary differential equations. However, in order to do this, we must first compute

(1) $P_N(\eta,\phi) \in Z_N$ for $(\eta,\phi) \in Z$

(2) $A_N(t)$, $t \geq s$,

with respect to the basis $\{\hat{\beta}_j^N\}_{j=1}^{k_N}$. We begin with (1).

Since P_N is the orthogonal projection $Z_\rho \to Z_N$, the orthogonality relationship in Z_ρ

$$\{P_N(\eta,\phi) - (\eta,\phi)\} \perp Z_N \tag{4.3}$$

uniquely determines $P_N(\eta,\phi)$, and therefore w_N^s as well. Expression (4.3) is equivalent to

$$\langle\hat{\beta}^N, \hat{\beta}^N w_N^s - (\eta,\phi)\rangle_\rho = 0,$$

which implies

$$<\hat{\beta}^N, \hat{\beta}^N w_N^s>_\rho = <\hat{\beta}^N, (\eta,\phi)>_\rho ,$$

or

$$Q_N w_N^s = h_N(\eta,\phi) , \tag{4.4}$$

where

$$Q_N = <\hat{\beta}^N, \hat{\beta}^N>_\rho = \beta^N(0)^T\beta^N(0) + \int_{-r}^{0} \beta^N(\theta)^T\beta^N(\theta)\rho(\theta)d\theta ,$$

and

$$h_N(\eta,\phi) = <\hat{\beta}^N, (\eta,\phi)>_\rho = \beta^N(0)^T\eta + \int_{-r}^{0} \beta^N(\theta)^T\phi(\theta)\rho(\theta)d\theta.$$

Therefore,

$$w_N^s = Q_N^{-1} h_N(\eta,\phi) .$$

The calculations above provide a means by which $F_N(t)$ can be computed as well. Indeed, (4.4) implies

$$Q_N F_N(t) = h_N(f(t),0),$$

but

$$h_N(f(t),0) = \beta^N(0)^T f(t),$$

and hence,

$$F_N(t) = Q_N^{-1}\beta^N(0)^T f(t).$$

We next address (2). Let $\hat{\phi}_N = (\phi_N(0),\phi_N) \in Z_N$ and suppose $\alpha_N \in R^{k_N}$ is such that

$$\hat{\phi}_N = \hat{\beta}^N \alpha_N .$$

Furthermore, for each $t \geq s$, let $\gamma_N(t)$ be such that

$$A_N(t)\hat{\phi}_N = \hat{\beta}^N \gamma_N(t).$$

It, of course, then follows that

$$\gamma_N(t) = A_N(t)\alpha_N. \tag{4.5}$$

Since $A_N(t)\hat{\phi}_N = P_N A(t)\hat{\phi}_N = P_N(L(t)\phi_N, D\phi_N)$, $\gamma_N(t)$ is the coordinate vector representation for $(L(t)\phi_N, D\phi_N)$. Therefore, by (4.4)

$$Q_N\gamma_N(t) = h_N(L(t)\phi_N, D\phi_N) = h_N(L(t)\beta^N\alpha_N, (D\beta^N)\alpha_N) = H_N(t)\alpha_N,$$

where

$$H_N(t) = h_N(L(t)\beta^N, (D\beta^N))$$

$$= \beta^N(0)^T(L(t)\beta^N) + \int_{-r}^{0} \beta^N(\theta)^T(D\beta^N)(\theta)\rho(\theta)d\theta .$$

Thus,

$$Q_N\gamma_N(t) = H_N(t)\alpha_N,$$

or

$$\gamma_N(t) = Q_N^{-1}H_N(t)\alpha_N,$$

which by (4.5) implies

$$A_N(t) = Q_N^{-1}H_N(t).$$

Since we have assumed that $\{Z_N, P_N, A_N(t)\}$ satisfies the hypotheses and conditions of Theorem 3.1, it follows that

$$\lim_{N\to\infty} \hat{\beta}^N w^N(t) = (x(t;\eta,\phi,f), x_t(\eta,\phi,f)),$$

and

$$\lim_{N\to\infty} \beta^N(0) w^N(t) = x(t;\eta,\phi,f) ,$$

uniformly in t for $t \in [s,T]$ and uniformly in f for f in bounded subsets of $L_2^n(s,T)$.

We conclude this section with the description of a particular approximation scheme which is included in the framework constructed in section 3. While the scheme we develop is closely related to the spline based schemes discussed in [6] and [8], slight differences in formulation necessitate the presentation of the scheme's development in detail.

Consider the partition of $[-r,0]$ given by $\{t_j^N\}_{j=0}^N$ with $t_j^N = -j\frac{r}{N}$ $j = 0,1,2,\cdots,N$, and define

$$Z_N^1 = \{(\eta,\phi) \in Z : \eta = \phi(0),\ \phi \text{ a first order spline function with knots at } t_j^N,\ j = 0,1,2,\cdots,N\}.$$

The set Z_N^1 is a finite dimensional subspace of Z with $\dim Z_N^1 = n(N+1)$. A basis for Z_N^1 may be constructed as follows:

For $\{t_j^N\}_{j=0}^N$ as above, and each $j = 0,1,2,\cdots,N$, let $e_j^N(\cdot):[-r,0] \to R$ denote the "hat" functions defined by

$$e_0^N(\theta) = \begin{cases} \frac{N}{r}(\theta - t_1^N) & t_1^N \le \theta \le 0 \\ 0 & \text{otherwise} \end{cases} ,$$

$$e_j^N(\theta) = \begin{cases} -\frac{N}{r}(\theta - t_{j-1}^N) & t_j^N \le \theta \le t_{j-1}^N \\ \frac{N}{r}(\theta - t_{j+1}^N) & t_{j+1}^N \le \theta \le t_j^N \\ 0 & \text{otherwise} \end{cases} , \quad j = 1,2,\cdots,N-1 ,$$

$$e_N^N(\theta) = \begin{cases} -\frac{N}{r}(\theta - t_{N-1}^N) & -r \le \theta \le t_{N-1}^N \\ 0 & \text{otherwise} \end{cases},$$

and let

$$\hat{\beta}_{(j+1)k}^N = (e_j^N(0)v_k, e_j^N v_k) \qquad j = 0,1,2,\cdots,N \;,\; k = 1,2,\cdots,n \;,$$

with $v_k = (0,\cdots,0,1,0,\cdots,0)^T \in R^n$ where the 1 appears in the *kth* position It can easily be verfied that $\{\hat{\beta}_\ell^N\}_{\ell=1}^{n(N+1)}$ is a basis for Z_N^1. Let P_N^1 be the orthogonal projection $Z_\rho \to Z_N^1$ and define $A_N^1(t) = P_N^1 A(t)$.

THEOREM 4.1. The approximation scheme $\{Z_N^1, P_N^1, A_N^1(t)\}$ satisfies all of the conditions in the statment of Theorem 3.1.

Proof. Since conditions (1) and (2) of Theorem 3.1 are trivially satisfied by $\{Z_N^1, P_N^1, A_N^1(t)\}$, we need only to argue that it satisfies condition (3) as well. That $\{Z_N^1, P_N^1, A_N^1(t)\}$ satisfies condition (3a) is established in the proof of Theorem 4.1 in [8]. Therefore we only discuss condition (3b) here.

Let $\hat{\phi} = (\phi(0),\phi) \in W$ with $\phi \in H^2(-r,0)$ and let $\hat{\phi}_N = P_N^1\hat{\phi} = (\phi_N(0),\phi_N)$. Theorem 2.5 of [22] (see also Theorem 21 of [23]) implies

$$(4.6) \qquad \int_{t_k^N}^{t_{k-1}^N} |D(\phi - \phi_N^I)|^2 \le \frac{1}{\pi^2}\left(\frac{r}{N}\right)^2 \int_{t_k^N}^{t_{k-1}^N} |D^2\phi|^2 \;,$$

and

$$(4.7) \qquad \int_{t_k^N}^{t_{k-1}^N} |\phi - \phi_N^I|^2 \le \frac{1}{\pi^4} \left(\frac{r}{N}\right)^4 \int_{t_k^N}^{t_{k-1}^N} |D^2\phi|^2 ,$$

where ϕ_N^I denotes the interpolatory spline for $\phi \in H^2(-r,0)$ with knots at $\{t_j^N\}_{j=0}^N$. From (4.6) and (4.7) we find

$$(4.8) \qquad |D(\phi - \phi_N^I)|_2 \le \frac{1}{\pi} \frac{r}{N} |D^2\phi|_2 ,$$

$$(4.9) \qquad |\phi - \phi_N^I|_2 \le \frac{1}{\pi^2} \left(\frac{r}{N}\right)^2 |D^2\phi|_2 .$$

Making use of the norm equivalence relation

$$|\cdot|_Z \le |\cdot|_\rho \le \sqrt{\nu}\ |\cdot|_Z ,$$

together with the minimality properties of the orthogonal projection $P_N^1 : Z_\rho \to Z_N^1$ we find

$$\begin{aligned}(4.10) \qquad |\phi_N - \phi|_2 &\le |P_N^1\hat{\phi} - \hat{\phi}|_Z = |\hat{\phi}_N - \hat{\phi}|_Z \\ &\le |\hat{\phi}_N - \hat{\phi}|_\rho \le |\hat{\phi}_N^I - \hat{\phi}|_\rho = |\phi_N^I - \phi|_{2,\rho} \\ &\le \sqrt{\nu}\ |\phi_N^I - \phi|_2 \le \frac{\sqrt{\nu}}{\pi^2} \left(\frac{r}{N}\right)^2 |D^2\phi|_2 ,\end{aligned}$$

where

$$\hat{\phi}_N^I \equiv (\phi_N^I(0), \phi_N^I) = (\phi(0), \phi_N^I) \in Z_N^1 .$$

We next use the Schmidt inequality [22] to estimate $|D(\phi^N - \phi_N^I)|_2$. Since ϕ_N, ϕ_N^I are linear on each sub-interval $[t_j^N, t_{j-1}^N]$ we have

$$(4.11) \quad |D(\phi_N - \phi_N^I)|_2^2 = \int_{-r}^{0} |D(\phi_N - \phi_N^I)|^2 = \sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} |D(\phi_N - \phi_N^I)|^2$$

$$\leq \sum_{j=1}^{N} \kappa \left(\frac{N}{r}\right)^2 \int_{t_j^N}^{t_{j-1}^N} |\phi_N - \phi_N^I|^2$$

$$\leq \sum_{j=1}^{N} \kappa \left(\frac{N}{r}\right)^2 \left\{ \int_{t_j^N}^{t_{j-1}^N} |\phi_N - \phi|^2 + \int_{t_j^N}^{t_{j-1}^N} |\phi - \phi_N^I|^2 \right\}$$

$$\leq \kappa \left(\frac{N}{r}\right)^2 \int_{-r}^{0} |\phi_N - \phi|^2 + \frac{\kappa}{\pi^4} \left(\frac{r}{N}\right)^2 \sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} |D^2\phi|^2$$

$$= \kappa \left(\frac{N}{r}\right)^2 |\phi_N - \phi|_2^2 + \frac{\kappa}{\pi^4} \left(\frac{r}{N}\right)^2 |D^2\phi|_2^2$$

$$\leq \kappa \left(\frac{N}{r}\right)^2 \frac{\nu}{\pi^4} \left(\frac{r}{N}\right)^4 |D^2\phi|_2^2 + \frac{\kappa}{\pi^4} \left(\frac{r}{N}\right)^2 |D^2\phi|_2^2$$

$$= (\nu + 1) \frac{\kappa}{\pi^4} \left(\frac{r}{N}\right)^2 |D^2\phi|_2^2 ,$$

where we have used (4.7) and (4.10) in making the estimates above. Therefore, by (4.9) and (4.11) we find

$$(4.12) \quad |D(\phi_N - \phi)|_2 \leq |D(\phi_N - \phi_N^I)|_2 + |D(\phi_N^I - \phi)|_2$$

$$\leq \frac{\sqrt{(\nu+1)\kappa}}{\pi^2} \left(\frac{r}{N}\right) |D^2\phi|_2 + \frac{1}{\pi^2} \left(\frac{r}{N}\right) |D^2\phi|_2$$

$$\leq \hat{K}(N) |D^2\phi|_2,$$

where

$$\hat{K}(N) = O(\frac{1}{N}).$$

Noting that $\hat{K}(N) \leq K$, K independent of N and $\phi \in H^2(-r,0)$, we have

$$|D(\phi_N - \phi)|_2 \to 0 \qquad N \to \infty,$$

and

$$|D(\phi_N - \phi)|_2 \leq K|D^2\phi|_2,$$

which establishes the second part of condition (3b). To see that for each t, $L(t)\phi_N \to L(t)\phi$, it can be argued (cf. [8], Theorem 4.1) that (4.12) implies

$$|\phi_N(\theta) - \phi(\theta)| \leq O(\frac{1}{N}),$$

as $N \to \infty$, uniformly in θ for $\theta \in [-r,0]$. Therefore, for each $t \geq s$, $|L(t)\phi_N - L(t)\phi| \leq O(\frac{1}{N})$ as $N \to \infty$ and condition (3b) has been established.

For $\{\hat{\beta}^N_\ell\}_{\ell=1}^{n(N+1)}$, the Z^1_N basis defined earlier, the matrices Q_N and $H_N(t)$ take on a particularly simple form. Indeed, for the case $\nu = 1$ (and therefore $\rho(\theta) = 1$) we have

$$Q_N^1 = \frac{r}{N} \begin{bmatrix} \frac{N}{r}+\frac{1}{3} & \frac{1}{6} & 0 & \cdots & & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \cdots & & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \otimes I \quad ,$$

and

$$H_N^1(t) = H_N^{11}(t) + H_N^{12},$$

where

$$H_N^{11}(t) = \begin{bmatrix} A_0(t) + D_0^N(t) & D_1^N(t) & \cdots & D_{N-1}^N(t) & A_1(t) + D_N^N(t) \\ 0 & 0 & \cdot & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

with

$$D_j^N(t) = \int_{-r}^{0} A(t,\theta)e_j^N(\theta)d\theta \ ,$$

and

$$H_N^{12} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \otimes I,$$

for $N = 2,3,\cdots$ where $\otimes$ denotes the Kronecker product and I is the $n \times n$ identity matrix.

As in the case of the approximation schemes developed for autonomous equations in [8], modifications of the results presented above can be used to verify that approximation schemes employing higher order spline functions and satisfying the conditions in the statement of Theorem 3.1 can be constructed.

5. Approximation Schemes Incorporating Time Discretization

In this section we briefly outline and discuss two alternative approximation scheme formulations based upon the approximation framework and state discretization techniques developed in section 3. However, the two schemes which we are going to introduce differ from the strictly semi-discrete schemes discussed previously in that these schemes incorporate various degrees of time discretization together with the state discretization. In particular, the first alternative allows for the discretization of the time dependence of the operator $L(t)$. This capability is extremely desirable when the time dependent matrix coefficients on the right hand side of the FDE are computationally expensive to evaluate. On the other hand, the second scheme allows for the complete discretization of all time dependence appearing in the equation. These schemes result in difference equation approximations and are in the same spirit as the approximation techniques discussed in [18] and [20]. We discuss each alternative separately and in turn.

Let $\{Z_N, P_N, A_N(t)\}$ be an approximation scheme for the initial value problem (2.1), (2.2) and for each $t \geq s$ define the operators $\hat{A}_N(t): Z_N \to Z_N$ by

$$(5.1) \qquad \hat{A}_N(t) = A_N(k\tfrac{r}{N}), \qquad (k-1)\tfrac{r}{N} < t \leq k\tfrac{r}{N}$$

for $k = i, i+1, \cdots$, where i is that integer for which $(i-1)\frac{r}{N} < s \leq i\frac{r}{N}$. We consider the ordinary differential equation in Z_N given by

$$(5.2) \qquad \dot{z}_N(t) = \hat{A}_N(t) z_N(t).$$

Since $\hat{A}_N(t)$ is piecewise constant, the evolution operator corresponding to (5.2) is of the following form:

If $(i-1)\frac{r}{N} < s \leq t \leq i\frac{r}{N}$,

$$\hat{U}_N(t,s) = \exp\left[(t-s)A_N(i\frac{r}{N})\right].$$

If $(i-1)\frac{r}{N} < s \leq i\frac{r}{N} < \cdots < j-1\frac{r}{N} < t \leq j\frac{r}{N}$,

$$\hat{U}_N(t,s) = \exp\left[(t-(j-1)\frac{r}{N})A_N(j\frac{r}{N})\right]\exp\left[\frac{r}{N}A_N((j-1)\frac{r}{N})\right]\cdots\cdots$$

$$\cdots\cdots\exp\left[\frac{r}{N}A_N((i+1)\frac{r}{N})\right]\exp\left[(i\frac{r}{N}-s)A_N(i\frac{r}{N})\right].$$

It then follows that

$$|\hat{U}_N(t,s)|_\rho \leq e^{\omega(t-(j-1)\frac{r}{N})} e^{\omega(i\frac{r}{N}-s)} \prod_{k=i+1}^{j-1} e^{\omega(\frac{r}{N})} = e^{(t-s)\omega},$$

and therefore for $t \leq T$

$$|\hat{U}_N(t,s)| \leq Me^{\omega(T-s)},$$

where

$$M = \sqrt{\nu}\ .$$

THEOREM 5.1. Consider the FDE initial value problem (2.1), (2.2) under the additional assumption stated in Theorem 3.1. Suppose further that $\{Z_N, P_N, A_N(t)\}$ is an approximation scheme satisfying conditions (1), (2), (3) of Theorem 3.1. Then if $\{Z_N, P_N, \hat{A}_N(t)\}$ is an approximation scheme with $\hat{A}_N(t)$ defined as in (5.1), we have

$$\lim_{N\to\infty} |[P_N U(t,s) - \hat{U}_N(t,s)P_N]z_0| = 0,$$

for each $z_0 \in Z$ uniformly in t for $t \in [s,T]$.

Proof. As in the proof of Theorem 3.1, it can be argued that for $z_0 \in D$

(5.3)
$$|\hat{U}_N(t,s)P_N z_0 - P_N U(t,s)z_0| = \left| \int_s^t \hat{U}_N(t,\tau)[\hat{A}_N(\tau)P_N - P_N A(\tau)]U(\tau,s)z_0 d\tau \right|$$
$$\leq Me^{\omega(T-s)} \int_s^T |[\hat{A}_N(\tau)P_N - P_N A(\tau)]z(\tau,s)| d\tau \ .$$

The desired result will clearly follow if we can demonstrate that for each $\tau \in [s,T]$ the integrand in the last expression in (5.3) above tends to zero as $N \to \infty$, with the convergence being dominated. Fortunately, however, this can be argued in precisely the same manner in which it was done in the proof of Theorem 3.1 with one minor exception. We must show that for $(j-1)\frac{r}{N} < \tau \leq j\frac{r}{N}$, $z(\tau,s) = (x(\tau),x_\tau)$ and $P_N z(\tau,s) \equiv (x^N(\tau),x_\tau^N)$,

$$L(j\tfrac{r}{N})x_\tau^N \to L(\tau)x_\tau \ ,$$

with the convergence being dominated. Using the estimates computed in proving the analogous claim in Theorem 3.1, we have

(5.4)
$$\begin{aligned} |L(j\tfrac{r}{N})x_\tau^N - L(\tau)x_\tau| &\leq |L(j\tfrac{r}{N})(x_\tau^N - x_\tau)| + |(L(j\tfrac{r}{N}) - L(\tau))x_\tau| \\ &\leq \left(\sum_{j=0}^{\nu} |A_j|_\infty + r^{\frac{1}{2}} \left| \left(\int_{-r}^0 |A(\cdot,\theta)|^2 d\theta \right)^{\frac{1}{2}} \right|_\infty \right) \\ &\qquad \cdot \left(|x^N - x|_\infty + r^{\frac{1}{2}} |D(x_\tau^N - x_\tau)|_2 \right) \\ &\quad + \left(\sum_{j=0}^{\nu} |A(j\tfrac{r}{N}) - A(\tau)| + \int_{-r}^0 |A(j\tfrac{r}{N},\theta) - A(\tau,\theta)| d\theta \right) |x_\tau| \\ &\leq M_1 \left(|x^N - x|_\infty + r^{\frac{1}{2}} |D(x_\tau^N - x_\tau)|_2 \right) \\ &\quad + M_2 \left(\sum_{j=0}^{\nu} |A(j\tfrac{r}{N}) - A(\tau)| + \int_{-r}^0 |A(j\tfrac{r}{N},\theta) - A(\tau,\theta)| d\theta \right) \\ &\to 0 \ \text{as} \ N \to \infty, \end{aligned}$$

where we have used conditions (3a) and (3b) to conclude that the first term above tends to zero and the continuity of the right hand side of the FDE to conclude that the second term tends to zero as well. Recalling that by condition (3b) we have

$$|D(x_\tau^N - x_\tau)|_2 \le K|D^2 x_\tau|_2 \le K|D^2 x|_2,$$

it then follows that the convergence in (5.4) is dominated.

Finally, as was the case in Theorem 3.1, D dense in Z and the operators $[\hat{U}_N(t,s)P_N - P_N U(t,s)]$ uniformly bounded are sufficient to guarantee convergence on all of Z and the theorem is proven.

We next turn our attention to the second alternative which involves the total discretization of the time dependency in the equation. Let $\{Z_N, P_N, A_N(t)\}$ be an approximation scheme for the initial value problem (2.1), (2.2) satisfying the hypotheses and conditions of Theorem 3.1. Suppose further that $\hat{A}_N(t)$ and i are as they were defined in (5.1). Then for each $N = 1,2,\cdots$, and each $t \ge s$ define the operator $V_N(t,s): Z_N \to Z_N$ by:

$$V_N(t,s) = \left(I - (t-s)\hat{A}_N(t)\right)^{-1}$$

if

$$(i-1)\frac{r}{N} < s \le t \le i\frac{r}{N}$$

and

$$V_N(t,s) = \left(I - (t-(j-1)\tfrac{r}{N})A_N(t)\right)^{-1}\left(I - \tfrac{r}{N}A_N((j-1)\tfrac{r}{N})\right)^{-1} \;\cdots\cdots$$

$$\cdots\cdots\; \left(I - \tfrac{r}{N}A_N((i+1)\tfrac{r}{N})\right)^{-1}\left(I - (i\tfrac{r}{N} - s)A_N(s)\right)^{-1}$$

if

$$(i-1)\frac{r}{N} < s \le i\frac{r}{N} < \cdots < (j-1)\frac{r}{N} < t \le j\frac{r}{N}.$$

In light of the arguments regarding the maximal dissipativeness of the operators $A_N(t) - \omega I$ given in the proof of Theorem 3.1, we have that for N sufficiently large, the operator inverses required in the definition of $V_N(t,s)$ above exist. Furthermore, using (3.1) it is easily verified that

$$|V_N(t,s)|_\rho \le \prod_{k=i}^{j} \frac{(\frac{N}{r})}{(\frac{N}{r}) - \omega} = (1 - \omega\,\frac{r}{N})^{-(j-i+1)} \le (1 - \omega\,\frac{r}{N})^{(t-s)\frac{N}{r} + 2} \to e^{\omega(t-s)} \quad \text{as } N \to \infty,$$

and hence, for $s \le t \le T$, $V_N(t,s)$ is uniformly bounded. In fact, it can be shown that for all N sufficiently large there exist an $\hat{\omega} > 0$ such that

(5.5) $$|V_N(t,s)| \le Me^{\hat{\omega}(T-s)},$$

where

$$M = \sqrt{\nu}.$$

THEOREM 5.2. Let $\{Z_N, P_N, A_N(t)\}$ be an approximation scheme for the initial value problem (2.1), (2.2) under the additional smoothness assumption stated in Theorem 3.1. Suppose further that $\{Z_N, P_N, A_N(t)\}$ satisfies conditions (1), (2), and (3) of that theorem. Then for $V_N(t,s)$ as defined above, we have

$$\lim_{N \to \infty} |[P_N U(t,s) - V_N(t,s)P_N]z_0| = 0$$

for each $z_0 \in Z$ uniformly in t for $t \in [s,T]$.

The proof of Theorem 5.2 can be argued in a manner similar to that used by Yosida in the proof of Theorem XIV.2 in [25]. We shall outline the essential ideas. Using the definition of $V_N(t,s)$ and the resolvent identity (i.e. for $\mathcal{A}$, $\mathcal{B}$ linear operators and $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{B})$ we have

$R(\lambda;A) - R(\lambda;B) = R(\lambda;A)(A-B)R(\lambda;B)$ it is not difficult to show that for N sufficiently large

$$\frac{\partial}{\partial s} V_N(t,s) = -V_N(t,s)\hat{A}_N(s)\left(I - (i\,\frac{r}{N} - s)\hat{A}_N(s)\right)^{-1}. \tag{5.6}$$

This in turn implies that for $z_0 \in D$

$$\begin{aligned}\frac{\partial}{\partial \tau}\,[V_N(t,\tau)P_N P_N U(\tau,s)z_0]
&= \left[\frac{\partial}{\partial \tau} V_N(t,\tau)P_N\right] P_N U(\tau,s)z_0 + V_N(t,\tau)P_N P_N \frac{\partial}{\partial \tau} U(\tau,s)z_0 \\
&= V_N(t,\tau)\left[P_N A(\tau) - \hat{A}_N(\tau)\left(I - (i\,\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} P_N\right] U(\tau,s)z_0.\end{aligned} \tag{5.7}$$

Integrating both sides of (5.7) from s to t we find

$$\begin{aligned}&[P_N U(t,s) - V_N(t,s)P_N]z_0 \\
&\quad = \int_s^t V_N(t,\tau)\left[P_N A(\tau) - \left(I - (i\,\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1}\hat{A}_N(\tau)P_N\right] U(\tau,s)z_0\,d\tau.\end{aligned}$$

Using the bound given in (5.5) we have for $z_0 \in D$ and $z(t,s) = U(t,s)z_0$

$$\begin{aligned}&|[P_N U(t,s) - V_N(t,s)P_N]z_0| \\
&\le \int_s^t |V_N(t,\tau)|\;\left|\left(I - (i\,\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1}\right|\;\left|\left[\hat{A}_N(\tau)P_N - P_N A(\tau)\right] z(\tau,s)\right| d\tau \\
&\quad + \int_s^t |V_N(t,\tau)|\;\left|\left[\left(I - (i\,\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} - I\right] P_N A(\tau)z(\tau,s)\right| d\tau \\
&\le M^2 e^{\hat{\omega}(T-s)}(1 - \omega\,\frac{r}{N})^{-1} \int_s^T |[\hat{A}_N(\tau)P_N - P_N A(\tau)]z(\tau,s)|\,d\tau\end{aligned}$$

$$+ \quad Me^{\hat{\omega}(T-s)} \int_s^T \left| \left| \left(I - (i\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} - I \right| P_N A(\tau) z(\tau,s) \right| d\tau$$

$$\leq \gamma_1 \int_s^T \left| \left| \hat{A}_N(\tau) P_N - P_N A(\tau) \right| z(\tau,s) \right| d\tau$$

$$+ \quad \gamma_2 \int_s^T \left| \left| \left(I - (i\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} P_N - P_N I \right| A(\tau) z(\tau,s) \right| d\tau$$

$$\equiv \mathcal{T}_1 + \mathcal{T}_2 .$$

We have already argued (cf. proof of Theorem 5.1) that $\mathcal{T}_1 \to 0$ as $N \to \infty$, uniformly in t for $t \in [s,T]$. Consider the term $\mathcal{T}_2$. Since

$$\left(I - (i\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} = P_{1,0}\left((i\frac{r}{N} - \tau)\hat{A}_N(\tau)\right),$$

where $P_{1,0}(z)$ denotes the $(1,0)$ entry in the Padé table of rational function approximations to the exponential we can apply [20, Theroem 10.3] and thus conclude that for each $\tau \in [s,T]$

$$\left| \left[\left(I - (i\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} P_N - P_N I \right] A(\tau) z(\tau,s) \right| \to 0$$

as $N \to \infty$. Furthermore, since $z_0 \in D$ and $z(\tau,s) = (x(\tau), x_\tau)$ we have

$$\left| \left| \left(I - (i\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} P_N - P_N I \right| A(\tau) z(\tau,s) \right|^2$$

$$\leq \left| \left[\left(I - (i\frac{r}{N} - \tau)\hat{A}_N(\tau)\right)^{-1} P_N - P_N I \right] \right|^2 \left| A(\tau) z(\tau,s) \right|^2$$

$$\leq \left(M(1 - \omega\frac{r}{N})^{-1} + 1\right)^2 \left| (L(\tau)x_\tau, Dx_\tau) \right|^2 \leq \left(M(1 - \omega\frac{r}{N})^{-1} + 1\right)^2 \left[\left| L(\tau)x_\tau \right|^2 + \left| Dx_\tau \right|^2 \right]$$

$$\leq \left(M(1 - \omega\frac{r}{N})^{-1} + 1\right)^2 \left[\left[\left(\sum_{j=0}^{\nu} |A_j|_\infty + r^{\frac{1}{2}} \left| \left(\int_{-r}^0 (A(\cdot,\theta))^2 d\theta \right)^{\frac{1}{2}} \right|_\infty \right) |x|_\infty \right]^2 + |Dx|^2_{L_2(s-r,T)} \right]$$

$$\leq M_0 ,$$

for N sufficiently large. Therefore $T_2 \to 0$ as $N \to \infty$ uniformly in t for $t \in [s,T]$ and hence

$$|[P_N U(t,s) - V_N(t,s)P_N]z_0| \to 0 \qquad \text{as} \quad N \to \infty ,$$

uniformly in t for $t \in [s,T]$ for each $z_0 \in D$. Since D is a dense subset of Z and the operators $[P_N U(t,s) - V_N(t,s)P_N]$ are uniformly bounded we once again can extend the convergence to all of Z and the theorem is proven.

If we define

$$\hat{z}_N(t,s;\eta,\phi,f) = \hat{U}_N(t,s)P_N(\eta,\phi) + \int_s^t \hat{U}_N(t,\tau)P_N(f(\tau),0)d\tau,$$

and

$$w_N(t,s;\eta,\phi,f) = V_N(t,s)P_N(\eta,\phi) + \int_s^t V_N(t,\tau)P_N(f(\tau),0)d\tau,$$

then all of the statements in Theorem 3.2 regarding the convergence of the approximations to the solution of the nonhomogeneous problem remain valid with z_N replaced by either $\hat{z}_N$ or w_N.

Remark. The operators $U_N(t,s)$ and $\hat{U}_N(t,s)$ defined previously must be computed indirectly. That is, they are computed as the numerical solution to the ordinary differential equation initial value problems in Z_N given by

$$\dot{U}_N(t,s) = A_N(t)U_N(t,s)$$
$$U_N(s,s) = I$$

and

$$\dot{\hat{U}}_N(t,s) = \hat{A}_N(t)\hat{U}_N(t,s),$$

$$\hat{U}_N(s,s) = I \ ,$$

respectively. The operator $V_N(t,s)$ on the other hand, can be computed directly. Although in practice we never actually compute the operators U_N, $\hat{U}_N$ or V_N, the above observation reveals that when one uses the fully discrete scheme, there is no further approximation required beyond the approximation of $A(t)$ by $A_N(t)$ and z_0 by $P_N z_0$. However, the time discretization employed in the definition of $V_N(t,s)$ is essentially a backwards Euler scheme which is only first order convergent (cf. [20]). It is unlikely, therefore, that the discrete scheme we have presented would perform as well as the semi-discrete schemes used in conjunction with say a fourth order Runge Kutta method to integrate the approximating ordinary differential equation. The discrete scheme should not however be totally discarded in that it does represent a starting point for the development of a discrete approximation framework (similar to the one developed for autonomous delay systems in [20]) encompassing time discretizations of arbitrarily high order.

6. Numerical Results

In this section we discuss a variety of numerical examples which demonstrate the feasibility of the approximation methods developed in sections 3 and 4. All computations were performed using a software package written for the IBM 370 model 158 computer at Brown University. Since our primary objective was to test our approximation methods and their convergence properties, factors such as computational efficiency and storage requirements were given only secondary consideration in the design of our programs. We note that software packages developed to implement the spline approximation schemes for linear autonomous functional differential equations discussed in [8] can easily be modified so as to be able to generate approximate solutions to nonautonomous equations via the schemes which we have presented above.

In all of the examples which follow, the approximation scheme employed represents a realization of the linear spline scheme discussed in section 4. In order to analyze the convergence properties of the scheme we have computed either exact solutions via the method of steps or highly accurate approximate solutions using the method of steps combined with a fourth order Runge Kutta integration scheme for ordinary differential equations. In the latter case, we emphasize that although our reference solutions are approximate, they were computed using methods which are completely independent of the approximation schemes which we are testing, and hence should not lead to invalid conclusions. In fact, for the examples for which exact solutions were available, we also computed approximate reference solutions and found them to be virtually indistinguishable from numerical tabulations of the exact solutions. The examples for which the exact solution was used as the reference solution are those for which the appropriate formulas have been provided.

The entries δ_N in the tables which follow represent the absolute differences between the reference solution x (exact or approximate) and our approximate solutions evaluated at sample points $\{t_k\}$ along the interval of interest. That is

$$\delta_N = |x_j(t_k) - [\beta^N(0)w^N(t_k)]_j|,$$

for $j = 1$ or 2, depending on the usage below, where $w^N(t) = (w_1^N(t),\cdots,w_{N+1}^N(t))^T$ is the solution vector of the approximating ordinary differential equation computed using a fourth order Runge Kutta method and β^N denotes the matrix function on $(-r,0)$ defined in terms of the linear spline basis given in section 4.

Example 6.1.

For our first example we consider the first order homogeneous equation

$$\dot{x}(t) = x(t) + tx(t-1),$$

on the interval $[0,2]$ with constant initial data given by

$$x_0(\theta) \equiv 1 , \qquad -1 \leq \theta \leq 0.$$

For this example we have computed the exact solution and it is given by

$$x(t) = \begin{cases} 2e^t - t - 1 & t \in [0,1] \\ 2e^t + (t^2 - 8)e^{t-1} + t^2 + 2t + 2 & t \in [1,2] \end{cases} .$$

Upon inspection of the numeri al results given in Table 6.1, it is easily seen that convergence is second order.

Table 6.1

t	$x(t)$	δ_2	δ_4	δ_8	δ_{16}	δ_{32}
0	1.0	0	0	0	0	0
.25	1.31805	.0310	.0108	.0024	.0006	.0002
.5	1.7974	.0655	.0167	.0043	.0010	.0002
.75	2.4840	.1079	.0275	.0069	.0017	.0004
1.0	3.43656	.1482	.0391	.0097	.0024	.0005
1.25	4.7773	.1933	.0519	.0145	.0036	.0009
1.5	6.73324	.2859	.0823	.0205	.0051	.0013
1.75	9.6190	.4560	.1282	.0325	.0081	.0020
2.0	13.9050	.7405	.2007	.0516	.0130	.0033

Example 6.2.

For our next example we consider the governing equation from an optimal control problem discussed in [18]

$$\dot{x}(t) = 6tx(t-1) + u^*(t) \qquad t \in [0,2]$$

$$x_0(\theta) \equiv 1 \qquad -1 \leq \theta \leq 0$$

$$u^*(t) = \begin{cases} -3\alpha t^2 - 6\alpha t + 10\alpha & t \in [0,1] \\ \alpha & t \in [1,2], \end{cases}$$

where $\alpha = -23.5/44.8$. The nonhomogeneous term u^* appearing in the equation above is the point in $L_2(0,2)$ where the functional

$$\Phi(u) = \tfrac{1}{2}x^2(2) + \tfrac{1}{2}\int_0^2 u^2(t)dt,$$

achieves its minimum subject to the equation above. The exact solution is given by

$$x(t) = \begin{cases} -\alpha t^3 + 3(1-\alpha)t^2 + 10\alpha t + 1 & t \in [0,1] \\ (-1.2\alpha)t^5 + 4.5t^4 + (26\alpha-12)t^3 \\ \quad +(12-36\alpha)t^2 + \alpha t + (16.2\alpha - .5) & t \in [1,2]\ . \end{cases}$$

The numerical results for this example can be found in Table 6.2. Once again, the convergence is essentially second order.

Table 6.2

t	$x(t)$	δ_4	δ_8	δ_{16}	δ_{32}
0	1.0	0	0	0	0
.25	-.01733	.0238	.0034	.0010	.0002
.5	-.41378	.0002	.0008	.0002	.0000
.75	-.14017	.0089	.0013	.0009	.0002
1.0	.85267	.1353	.0405	.0122	.0039
1.25	1.43499	.1222	.0095	.0048	.0004
1.5	.7359	.1716	.0082	.0042	.0013
1.75	-.2029	.3263	.0670	.0125	.0033
2.0	.52455	.2170	.0657	.0186	.0055

Example 6.3.

Here we consider the scalar second order homogeneous equation

$$\ddot{x}(t) + t\dot{x}(t-1) + x(t) = 0.$$

On the interval $[0,3]$ with initial conditions given by

$$\left.\begin{aligned} x_0(\theta) &= \cos(\theta+1) \\ \dot{x}_0(\theta) &= -\sin(\theta+1) \end{aligned}\right. \qquad -1 \le \theta \le 0 .$$

Rewriting the initial value problem above as a first order system we have

$$\dot{y}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y(t) + \begin{bmatrix} 0 & 0 \\ 0 & -t \end{bmatrix} y(t-1),$$

$$y_0(\theta) = \begin{bmatrix} \cos(\theta+1) \\ -\sin(\theta+1) \end{bmatrix} \qquad -1 \le \theta \le 0,$$

where

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} .$$

Our numerical results for $y_1 = x$ and $y_2 = \dot{x}$ are tabulated in Tables 6.3 and 6.4 respectively. For this two dimensional example, the convergence is observed to be quadratic for both components of the solution.

Table 6.3

t	$y_1(t)$	δ_2	δ_4	δ_8	δ_{16}
0	.54030	.00168	.0003	.0000	.0000
.5	.075816	.0046	.0009	.0002	.0001
1.0	-.34086	.0277	.0076	.0019	.0005
1.5	-.46758	.0637	.0163	.0041	.0009
2.0	-.14448	.0885	.0211	.0051	.0012
2.5	.487092	.0546	.0092	.0018	.0003
3.0	.86544	.0868	.0291	.0079	.0022

Table 6.4

t	$y_2(t)$	δ_2	δ_4	δ_8	δ_{16}
0	-.84147	.0021	.0003	.0000	.0000
.5	-.95738	.0126	.0035	.0009	.0002
1.0	-.62366	.0247	.0060	.0015	.0003
1.5	.17833	.0221	.0050	.0011	.0002
2.0	1.07236	.0198	.0081	.0024	.0008
2.5	1.2591	.1283	.0369	.0096	.0025
3.0	.01298	.2552	.0447	.0085	.0218

Example 6.4.

We again consider a damped oscillator with delayed damping. In this example, however, we place the time varying coefficient in front of the restoring term instead of the damping term as was the case in the previous example. The initial value problem is given by

$$\ddot{x}(t) + \dot{x}(t-1) + tx(t) = 0 \qquad t \in [0,3] ,$$

$$\left.\begin{aligned} x_0(\theta) &= \cos(\theta+1) \\ \dot{x}_0(\theta) &= -\sin(\theta+1) \end{aligned}\right\} \qquad -1 \le \theta \le 0,$$

or

$$\dot{y}(t) = \begin{bmatrix} 0 & 1 \\ -t & 0 \end{bmatrix} y(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} y(t-1) \qquad t \in [0,3] ,$$

$$y_0(\theta) = \begin{bmatrix} \cos(\theta+1) \\ -\sin(\theta+1) \end{bmatrix} \qquad -1 \le \theta \le 0 ,$$

where

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} ,$$

if written as an equivalent first order system. Examination of the numerical results contained in Tables 6.5 and 6.6 reveals that the quadratic convergence is unaffected by the placement of the time varying coefficient. Once again we have second order convergence in both the x and $\dot{x}$ components.

Table 6.5					
t	$y_1(t)$	δ_2	δ_4	δ_8	δ_{16}
0	.540302	.0017	.0002	.0000	.0000
.5	.133213	.0121	.0032	.0008	.0002
1.0	-.166604	.0370	.0096	.0024	.0006
1.5	-.230397	.0605	.0152	.0037	.0009
2.0	-.035350	.0480	.0098	.0023	.0005
2.5	.262053	.0240	.0085	.0023	.0006
3.0	.365969	.1226	.0307	.0076	.0019

Table 6.6					
t	$y_2(t)$	δ_2	δ_4	δ_8	δ_{16}
0	-.841470	.0021	.0003	.0000	.0000
.5	-.752190	.0184	.0044	.0012	.0003
1.0	-.396825	.0213	.0062	.0016	.0004
1.5	.149976	.0069	.0050	.0012	.0004
2.0	.577757	.0884	.0225	.0056	.0015
2.5	.504380	.1601	.0384	.0094	.0024
3.0	-.163431	.0899	.0148	.0029	.0006

Example 6.5.

In this example we consider a damped oscillator with sinusoidal external force and delayed restoring force having a time varying coefficient. The initial value problem is given by

$$\ddot{x}(t) + \dot{x}(t) + tx(t-1) = \sin t ,$$

$$x_0(\theta) = \sin(\theta+1)$$
$$\dot{x}_0(\theta) = \cos(\theta+1) \qquad -1 \le \theta \le 0,$$

or

$$\dot{y}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} y(t) + \begin{bmatrix} 0 & 0 \\ -t & 0 \end{bmatrix} y(t-1) + \begin{bmatrix} 0 \\ \sin t \end{bmatrix},$$

$$y_0(\theta) = \begin{bmatrix} \sin(\theta+1) \\ \cos(\theta+1) \end{bmatrix} \qquad -1 \le \theta \le 0,$$

where

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}.$$

As the numerical results in Tables 6.7 and 6.8, substantiate, second order convergence in both the x and $\dot{x}$ components of the solution is maintained even with the inclusion of the nontrivial forcing term. However, as will become apparent from the numerical results discussed in Example 6.9, the rate of convergence is sensitive to the smoothness of the function appearing as the nonhomogeneous term in the equation.

Table 6.7

t	$y_1(t)$	δ_2	δ_4	δ_8	δ_{16}
0	.841470	.0021	.0003	.0000	.0000
.5	1.067646	.0006	.0007	.0001	.0000
1.0	1.243813	.0125	.0027	.0007	.0002
1.5	1.341552	.0152	.0040	.0010	.0003
2.0	1.269265	.0057	.0006	.0001	.0000
2.5	.902306	.0147	.0049	.0012	.0003
3.0	.132715	.0425	.0115	.0029	.0007

Table 6.8					
t	$y_2(t)$	δ_2	δ_4	δ_8	δ_{16}
0	.540302	.0017	.0002	.0000	.0000
.5	.395912	.0174	.0034	.0008	.0002
1.0	.296513	.0098	.0029	.0008	.0002
1.5	.063469	.0033	.0007	.0004	.0001
2.0	-.395815	.0115	.0047	.0009	.0002
2.5	-1.111002	.0238	.0054	.0014	.0003
3.0	-1.97221	.0355	.0091	.0021	.0005

Example 6.6.

For the second order equation

$$\ddot{x}(t) + e^{-t}\dot{x}(t-1) + x(t) = \sin t \qquad t \in [0,3] ,$$

with initial data

$$\begin{aligned} x_0(\theta) &= \cos(\theta+1) \\ \dot{x}_0(\theta) &= -\sin(\theta+1) \end{aligned} \qquad -1 \le \theta \le 0 ,$$

the numerical results given in Tables 6.9 and 6.10 exhibit second order convergence.

Table 6.9

t	$x(t)$	δ_2	δ_4	δ_8	δ_{16}
0	.540302	.0017	.0002	.0000	.0000
.5	.106974	.0093	.0026	.0007	.0002
1.0	-.171978	.0413	.0108	.0027	.0007
1.5	-.134712	.0671	.0167	.0041	.0010
2.0	.217952	.0554	.0123	.0029	.0007
2.5	.745662	.0035	.0033	.0010	.0003
3.0	1.227099	.0904	.0239	.0061	.0016

Table 6.10

t	$\dot{x}(t)$	δ_2	δ_4	δ_8	δ_{16}
0	-.841470	.0021	.0003	.0000	.0000
.5	-.791286	.0217	.0061	.0015	.0004
1.0	-.269420	.0232	.0050	.0012	.0003
1.5	.416949	.0127	.0039	.0011	.0003
2.0	.944983	.0766	.0206	.0052	.0013
2.5	1.08968	.1262	.0308	.0077	.0019
3.0	.756316	.1253	.0284	.0069	.0017

Example 6.7.

In this example we consider the same damped oscillator as the one discussed in Example 6.5. Here, however, we exclude the external forcing function and provide discontinuous initial data. The initial value problem is given by

$$\ddot{x}(t) + \dot{x}(t) + tx(t-1) = 0 \qquad t \in [0,3],$$

$$x_0(\theta) = \begin{cases} 1 & \theta \in [-1,-\frac{1}{2}] \\ -1 & \theta \in (-\frac{1}{2},0] \end{cases} ,$$

$$\dot{x}_0(\theta) = 0.$$

The numerical results for this example can be found in Tables 6.11 and 6.12. From the data contained in these tables, it can easily be inferred that the rate of convergence is sensitive to the smoothness of the function given as initial data. Indeed the convergence exhibited in both the x and $\dot{x}$ components of the solution is clearly not second order. It is interesting to note initial data that is continuous but not C^1 can also lead to sub-quadratic convergence. We shall see this in the next example.

Table 6.11

t	x(t)	δ_2	δ_4	δ_8	δ_{16}
0	-1.000	.0430	.0070	.0003	.0000
.5	-1.018469	.0094	.0324	.0260	.0069
1.0	-.994824	.0482	.0549	.0211	.0013
1.5	-.779472	.0701	.0112	.0137	.0010
2.0	-.344241	.1063	.0615	.0164	.0004
2.5	.304900	.1421	.0257	.0064	.0041
3.0	1.063963	.1508	.0018	.0028	.0059

Table 6.12

t	$\dot{x}(t)$	δ_2	δ_4	δ_8	δ_{16}
0	0.0	0	0	0	0
.5	-.106530	.0766	.0442	.0252	.0143
1.0	.228874	.0286	.0349	.0114	.0120
1.5	.641839	.0407	.0411	.0263	.0033
2.0	1.099665	.0880	.0161	.0167	.0051
2.5	1.463359	.0484	.0838	.0074	.0034
3.0	1.494070	.0533	.0387	.0018	.0066

Example 6.8.

Here we consider the second order equation

$$\ddot{x}(t) + t\dot{x}(t-1) + x(t) = 0 \qquad t \in [0,3],$$

with continuous but not C^1 initial data

$$x_0(\theta) = \begin{cases} 1+\theta & \theta \in [-1,-\tfrac{1}{2}] \\ -\theta & \theta \in [-\tfrac{1}{2}, 0] \end{cases},$$

$$\dot{x}_0(\theta) = \begin{cases} 1 & \theta \in [-1,-\tfrac{1}{2}] \\ -1 & \theta \in (-\tfrac{1}{2}, 0] \end{cases}.$$

Written as a first order system, the initial value problem becomes

$$\dot{y}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y(t) + \begin{bmatrix} 0 & 0 \\ 0 & -t \end{bmatrix} y(t-1),$$

$$y_0(\theta) = \begin{cases} \begin{bmatrix} 1+\theta \\ 1 \end{bmatrix} & \theta \in [-1,-\tfrac{1}{2}] \\ \begin{bmatrix} -\theta \\ -1 \end{bmatrix} & \theta \in (-\tfrac{1}{2},0] \end{cases},$$

with

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

It is evident from Tables 6.13 and 6.14 that second order convergence has not been achieved. In fact there does not appear to be a discernible pattern to the convergence. It can be pointed out that the initial data for this example is contained in Z_N^1, $N \geq 2$, where Z_N^1 denotes the linear spline approximation spaces being used (see section 4). It is evident from this example that this feature need not have a positive effect upon the performance of the approximation schemes. This is in contrast to the behavior observed for the averaging schemes discussed in [8] when applied to autonomous equations.

Table 6.13

t	$y_1(t)$	δ_2	δ_4	δ_8	δ_{16}
0	0.0	0.0	0.0	0.0	0.0
.5	-.5	.0937	.0361	.0122	.0045
1.0	-.843932	.2110	.0138	.0149	.0113
1.5	-.736574	.3184	.0264	.0263	.0202
2.0	-.123418	.2915	.0013	.0201	.0188
2.5	.629872	.0372	.0027	.0108	.0074
3.0	.756091	.3846	.0384	.0001	.0105

Table 6.14

t	$y_2(t)$	δ_2	δ_4	δ_8	δ_{16}
0	-1.0	.0819	.0122	.0005	.0000
.5	-1.0	.0735	.0717	.0307	.0214
1.0	-.289816	.1923	.0159	.0032	.0159
1.5	.752107	.0427	.0296	.0007	.0091
2.0	1.57099	.2871	.0718	.0383	.0180
2.5	1.172884	.6506	.0309	.0580	.040
3.0	-.920864	.6728	.0716	.0309	.0239

Example 6.9.

For our final example we again consider the damped oscillator with delayed restoring force of Example 6.5. Here, however, we include a discontinuous *external forcing function in the* equation. The initial value problem is given by

$$\ddot{x}(t) + \dot{x}(t) + tx(t-1) = 10u_{.5}(t) \qquad t \in [0,3],$$

$$\begin{aligned} x_0(\theta) &= \cos(\theta+1) \\ x_0(\theta) &= -\sin(\theta+1) \end{aligned} \qquad -1 \le \theta \le 0,$$

where $u_{.5}$ denotes the unit step at $t = .5$ defined by

$$u_{.5}(t) = \begin{cases} 0 & t < .5 \\ 1 & .5 \le t. \end{cases}$$

Rewritten as a first order system, the above problem becomes

$$\dot{y}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} y(t) + \begin{bmatrix} 0 & 0 \\ -t & 0 \end{bmatrix} y(t-1) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u_{.5}(t) ,$$

$$y_0(\theta) = \begin{bmatrix} \cos(\theta+1) \\ -\sin(\theta+1) \end{bmatrix} \qquad -1 \leq \theta \leq 0,$$

where

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} .$$

Not unexpectedly, we found that the observed rate of convergence for our approximation schemes was sensitive to the discontinuity in the forcing function. As the numerical results in Tables 6.15 and 6.16 bear out, convergence in the first component of the solution appears to be second order throughout, while in the case of the $\dot{x}$ component, quadratic convergence is obtained at some of the sample points but not at others. The discontinuity in the nonhomogeneous term introduces a jump in the second derivative of the solution. This, in turn, will degrade the performance of the approximation schemes.

Table 6.15

t	$y_1(t)$	δ_2	δ_4	δ_8	δ_{16}
0	.540302	.0017	.0002	.0000	.0000
.5	.191454	.0182	.0044	.0010	.0003
1.0	.982577	.1672	.0454	.0108	.0027
1.5	3.330325	.2486	.0585	.0148	.0037
2.0	6.631893	.2137	.0551	.0133	.0031
2.5	10.105261	.0876	.0077	.0017	.0001
3.0	12.380963	.2126	.0654	.0177	.0050

Table 6.16					
t	$y_2(t)$	δ_2	δ_4	δ_8	δ_{16}
0	-.841470	.0021	.0003	.0000	.0000
.5	-.557825	.0214	.0043	.0023	.0010
1.0	3.386594	.1169	.0115	.0047	.0012
1.5	5.823277	.0522	.0022	.0010	.0004
2.0	7.133736	.0543	.0261	.0104	.0036
2.5	6.298532	.2048	.0628	.0143	.0041
3.0	2.180855	.4984	.1208	.0318	.0086

Although a complete characterization of the convergence properties of the approximation schemes we have developed would be extremely difficult if not impossible to obtain, based on the numerical evidence which we have presented, the following conclusion can safely be drawn. Our schemes appear to yield quadratic convergence for initial value problems in which the initial data, external forcing function, coefficients and therefore the solution are smooth. On the other hand, it is a relatively simple matter to break the second order convergence in some or all of the components of the solution through the introduction of irregularity into the solution.

REFERENCES

[1] H. T. Banks, Approximation of nonlinear functional differential equation control systems, *J. Optimization Theory Appl.*, 29 (1979), 383 - 408.

[2] H. T. Banks, Identification of nonlinear delay systems using spline methods, *Proc. International Conf. on Nonlinear Phenomena in the Math Sciences*, Univ. Texas, Arlington, June 16 - 20, 1980, Academic Press, to appear.

[3] H. T. Banks, Parameter identification techniques for physiological control systems, *Lectures in Applied Math*, Vol. 19, Amer. Math. Society, Providence, R.I., 1981,

[4] H. T. Banks and J. A. Burns, An abstract framework for approximate solutions to optimal control problems governed by hereditary systems, *Intl. Conf. on Differential Equations*, (H. Antosiewicz, ed.), Academic Press, 1975, 10 - 25.

[5] H. T. Banks and J. A. Burns, Hereditary control problems: numerical methods based on averaging approximations, *SIAM J. Control and Optimization*, 16 (1978), 169 - 208.

[6] H. T. Banks, J. A. Burns and E. M. Cliff, Parameter estimation and identification for systems with delays, *SIAM J. Control and Optimization*, 1981, to appear.

[7] H. T. Banks and P. L. Daniel, Estimation of delays and other parameters in nonlinear functional differential equations, to appear.

[8] H. T. Banks and F. Kappel, Spline approximations for functional differential equations, *J. Differential Equations*, 34 (1979), 496 - 522.

[9] H. T. Banks and I. G. Rosen, Approximation techniques for parameter estimation in hereditary control systems, *Proc. IEEE - Conf. on Decision and Control* (Alberquerque, Dec., 1980), 741 - 743.

[10] P. L. Daniel, Spline-based approximation methods for the identification and control of nonlinear functional differential equations, *Ph.D. thesis*, Brown Univ., Providence, R. I., June, 1981.

[11] M. C. Delfour, The linear quadratic optimal control problem for hereditary differential systems: theory and numerical solution, *Appl. Math. Optim.*, 3 (1977), 101-162.

[12] J. S. Gibson, Linear quadratic optimal control of hereditary differential systems: infinite dimensional Riccati equations and numerical approximations, preprint, Sept., 1980, submitted to *SIAM J. Control and Optimization*.

[13] F. Kappel and W. Schappacher, Nonlinear functional differential equations and abstract integral equations, *Proc. Royal Soc. Edinburgh*, 84A (1979), 71 - 91.

[14] S. G. Krein, Linear Differential Equations in Banach Space, *Transl. Math. Monograph*, Vol. 29, Amer. Math. Soc., 1971, Providence, R. I.

[15] K. Kunisch, Approximation schemes for nonlinear neutral optimal control systems, preprint, May, 1980, to appear in *J. Math. Anal. Appl.*

[16] K. Kunisch, Approximation schemes for the linear quadratic optimal control problem associated with delay equations, *SIAM J. Control and Optimization*, to appear.

[17] D. C. Reber, Approximation and Optimal Control of Linear Hereditary Systems, *Ph.D. Thesis*, Brown Univ., Providence, R. I., November, 1977.

[18] D. C. Reber, A finite difference technique for solving optimization problems governed by linear functional differential equations, *J. Differential Equations*, 32 (1979), 192 - 232.

[19] I. G. Rosen, A discrete Approximation Framework for Hereditary Systems, *Ph.D. Thesis*, Brown Univ., Providence, R. I., June, 1980.

[20] I. G. Rosen, A discrete approximation framework for hereditary systems, *J. Differential Equations*, 40 (1981), 377-449.

[21] I. G. Rosen, Discrete approximation methods for parameter identification in delay systems, to appear.

[22] M. H. Schultz, Spline Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1973.

[23] M. H. Schultz and R. S. Varga, L-splines, *Numer. Math.*, 10 (1967), 345 - 369.

[24] K. C. Wei, W. N. Bianco, and M. M. Vikmanis, An anti-aircraft gunner model with delayed tracer measurements, *Proc. 1980 JACC*, August 13 - 15, San Francisco.

[25] K. Yosida, Functional Analysis, Springer Verlag, New York, 1974.